Catenary Curve

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1 Designing catenary curve

Based on Webster dictionary, a catenary is the curve assumed by a cord of uniform density and cross section that is perfectly flexible but not capable of being stretched and that hangs freely from two fixed points. In real world, we can find many examples of catenary as shown in the following images.

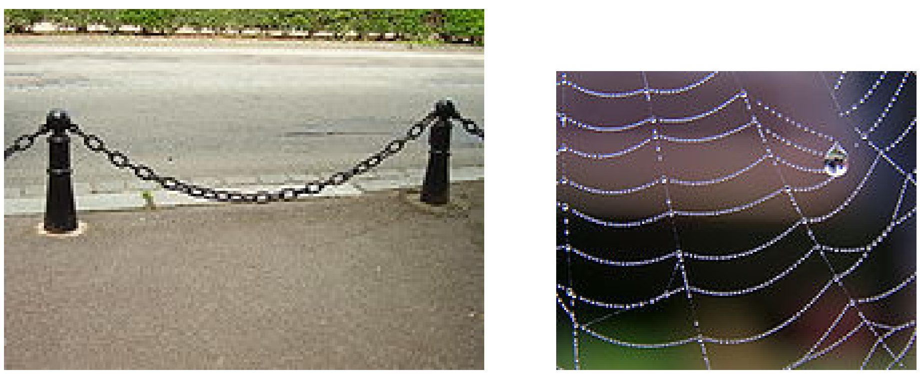

Figure 1: Chains and spider webs

Figure 2: Power lines

Mathematically, a catenary curve is the graph of the hyperbolic cosine function. Through a proper transformation, the equation of a catenary in Cartesian coordinate has the following standard simple representation:

$$y(x) = u\cosh\left(\frac{x}{u}\right) = \frac{u}{2}\left(e^{x/u} + e^{-x/u}\right),$$

where cosh is the hyperbolic cosine function. The above function is symmetric to y axis with $x = 0$ being the lowest point. The larger the u increases, the flatter the shape of curve becomes.

In real world application, we need to model a catenary curve with non-symmetric information. For example, we need to generate a catenary curve that passes through two poles of different heights. For this reason, we represent the catenary curve in the following generic form[1]:

$$y(x) = u\cosh\left(\frac{x-v}{u}\right) + \beta$$

where u, v, and β are three constants to be determined by the boundary conditions of the problem. Usually these conditions include two points from which the cable is being suspended and the length of the cable.

Assume the two points are given by $\mathbf{p}_1 = (x_1, y_1)$ and $\mathbf{p}_2 = (x_2, y_2)$. Substituting them into the above equation we have

$$y_1 = u\cosh\left(\frac{x_1-v}{u}\right) + \beta \tag{1.1}$$

$$y_2 = u\cosh\left(\frac{x_2-v}{u}\right) + \beta \tag{1.2}$$

In addition to the above two equations, we will need one more equation to form a system of three equations in three unknowns. We further assume that the cable length L is given. Accordingly, the third equation would be the arc length of catenary curve, i.e.,

$$L = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2}dx.$$

Since

$$y'(x) = u\sinh\left(\frac{x-v}{u}\right)\left(\frac{1}{u}\right) = \sinh\left(\frac{x-v}{u}\right),$$

we have

$$\begin{aligned} L &= \int_{x_1}^{x_2} \sqrt{1 + \sinh^2\left(\frac{x-v}{u}\right)}\,dx = \int_{x_1}^{x_2} \cosh\left(\frac{x-v}{u}\right)dx \\ &= u\int_{x_1}^{x_2} \cosh\left(\frac{x-v}{u}\right) d\left(\frac{x-v}{u}\right) = u\sinh\left(\frac{x-v}{u}\right)\Bigg|_{x_1}^{x_2} \\ &= u\left[\sinh\left(\frac{x_2-v}{u}\right) - \sinh\left(\frac{x_1-v}{u}\right)\right] \end{aligned} \tag{1.3}$$

Equations (2.1), (2.2), and (1.3) form a system of nonlinear equations. Solution to the three unknowns is non-trivial. In this document we illustrate how Newton's method can be used to solve the catenary problem. Let's rewrite Equations (2.1), (2.2), and (1.3) in the following forms:

$$f_1 = u\cosh\left(\frac{x_1 - v}{u}\right) + \beta - y_1 = 0$$

$$f_2 = u\cosh\left(\frac{x_2 - v}{u}\right) + \beta - y_2 = 0$$

$$f_3 = u\left[\sinh\left(\frac{x_2 - v}{u}\right) - \sinh\left(\frac{x_1 - v}{u}\right)\right] - L = 0$$

It is known that Newton's method requires to compute Jacobian matrix that consists of first order partial derivatives. Therefore, we need to compute the following derivatives

$$\frac{\partial f_1}{\partial u} = \cosh\left(\frac{x_1 - v}{u}\right) - \frac{x_1 - v}{u}\sinh\left(\frac{x_1 - v}{u}\right)$$

$$\frac{\partial f_1}{\partial v} = -\sinh\left(\frac{x_1 - v}{u}\right)$$

$$\frac{\partial f_1}{\partial \beta} = 1$$

$$\frac{\partial f_2}{\partial u} = \cosh\left(\frac{x_2 - v}{u}\right) - \frac{x_2 - v}{u}\sinh\left(\frac{x_2 - v}{u}\right)$$

$$\frac{\partial f_2}{\partial v} = -\sinh\left(\frac{x_2 - v}{u}\right)$$

$$\frac{\partial f_3}{\partial \beta} = 1$$

$$\frac{\partial f_3}{\partial u} = \sinh\left(\frac{x_2 - v}{u}\right) - \sinh\left(\frac{x_1 - v}{u}\right) - \frac{1}{u}\left[(x_2 - v)\cosh\left(\frac{x_2 - v}{u}\right) - (x_1 - v)\cosh\left(\frac{x_1 - v}{u}\right)\right]$$

$$\frac{\partial f_3}{\partial v} = \cosh\left(\frac{x_1 - v}{u}\right) - \cosh\left(\frac{x_2 - v}{u}\right)$$

$$\frac{\partial f_3}{\partial \beta} = 0$$

Let J denote the Jacobian matrix of the following form

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial \beta} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial \beta} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial \beta} \end{pmatrix}$$

Then, Newton iteration is done as follows

$$\begin{pmatrix} u^{(k)} \\ v^{(k)} \\ \beta^{(k)} \end{pmatrix} = \begin{pmatrix} u^{(k-1)} \\ v^{(k-1)} \\ \beta^{(k-1)} \end{pmatrix} - J^{-1} \times \begin{pmatrix} f_1\left(u^{(k-1)}, v^{(k-1)}\right) \\ f_2\left(u^{(k-1)}, v^{(k-1)}\right) \\ f_3\left(u^{(k-1)}, v^{(k-1)}\right) \end{pmatrix} \qquad (k = 1, 2, \cdots) \tag{1.4}$$

where J^{-1} is the inverse of Jacobian matrix evaluated at $(u^{(k-1)}, v^{(k-1)}, \beta^{(k-1)})$. We terminate the iteration when h are smaller than the given distance tolerance. It is readily seen that the lowest point is given by

$$\mathbf{P}_{\min} = (v, u + \beta)$$

The advantage of Newton method for solving systems of nonlinear equations is its quadratic convergence speed once a sufficient accurate approximation is known. A weakness of this method is that it may diverge when initial poor approximations of u, v, and β are given. To overcome such limitation, the Steepest Descent method may be implemented to find the good approximations of u, v, and β for Newton iteration method. The Steepest Descent method has only linear convergence but will usually converge even for poor initial approximations.

Two examples are given here for interested readers:

Example 1 (Symmetric case): With $\mathbf{p}_1 = (x_1, y_1) = (-50, 100), \mathbf{p}_2 = (x_2, y_2) = (50, 100), L = 150$, we take $u = L/3$, $v = (x_1 + x_2)/2$, and $\beta = 0.7 \times \min(y_1, y_2)$ as initial approximations and solve (1.4) for u, v, and β. After 7 iteration we obtain $u = 39.11697$, $v = 0.0$ and $\beta = 15.41196$ that guarantee f_1, f_2, and f_3 are all less than $1.0e^{-8}$. We can further derive $\mathbf{p}_{min} = (0.0, 54.5289)$.

Example 2 (Non-symmetric case): With $\mathbf{p}_1 = (x_1, y_1) = (-50, 100), \mathbf{p}_2 = (x_2, y_2) = (60, 120), L = 150$, we take again $u = L/3$, $v = (x_1 + x_2)/2$, and $\beta = 0.7 \times \min(y_1, y_2)$ as initial approximations and solve (1.4) for u, v, and β. After 7 iteration we obtain $u = 39.72898$, $v = -0.32893$ and $\beta = 24.95907$ that guarantee f_1, f_2, and f_3 are all less than $1.0e^{-8}$. We can further derive $\mathbf{p}_{min} = (-0.32893, 64.68805)$.

2 Catenary curve with clearance specified

In the previous section we discussed how to design a catenary curve based given two points and the cable length. If users' input is different, one has to derive a new set of equations to solve the three unknowns that meet the given conditions. For example, cables may drop lower during the frozen winter as illustrated in the following image. Accordingly, utility workers may want to set the minimum clearance of power lines for safety reason. Assume we are given coordinates of two

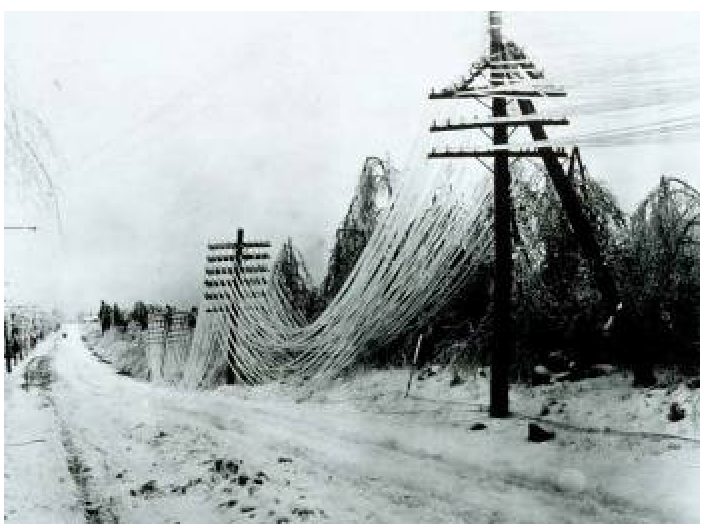

Figure 3: Due to added weight frozen cables are dragged much lower

poles and the lowest clearance height H. In this case, equations Equations (1.1) and (1.2) still hold. We need to derive the third equation to solve the three unknowns u, v, and β. Since the lowest point occurs at the end points or a local extrema at which

$$y'(x) = u \sinh\left(\frac{x-v}{u}\right)\left(\frac{1}{u}\right) = \sinh\left(\frac{x-v}{u}\right) = 0.$$

This in turn means that

$$\frac{x-v}{u} = 0, \Longleftrightarrow x = v.$$

Accordingly, we have

$$y(x) = u \cosh\left(\frac{x-v}{u}\right) + \beta \overset{\text{x=v}}{\Longrightarrow} H = u + \beta.$$

Replacing $\beta = H - u$ in equations (1.1) and (1.2) we obtain two equations for two unknowns:

$$y_1 = u \cosh\left(\frac{x_1 - v}{u}\right) + H - u \tag{2.1}$$

$$y_2 = u \cosh\left(\frac{x_2 - v}{u}\right) + H - u \tag{2.2}$$

Let

$$f(u, v) = u \cosh\left(\frac{x_1 - v}{u}\right) + H - y_1 - u = 0$$

$$g(u, v) = u \cosh\left(\frac{x_2 - v}{u}\right) + H - y_2 - u = 0$$

Differentiating f and g with respect to u and v gives

$$f_u = \cosh\left(\frac{x_1 - v}{u}\right) - \frac{x_1 - v}{u} \sinh\left(\frac{x_1 - v}{u}\right) - 1$$

$$f_v = \sinh\left(\frac{x_1 - v}{u}\right)$$

$$g_u = \cosh\left(\frac{x_2 - v}{u}\right) - \frac{x_2 - v}{u} \sinh\left(\frac{x_2 - v}{u}\right) - 1$$

$$g_v = \sinh\left(\frac{x_2 - v}{u}\right)$$

Accordingly, we can use equation (1.4) to solve u and v, noting that the third unknown is given by $\beta = H - u$.

3 Catenary curve with sag specified

Some applications require to restrict the sag at the middle between two poles as shown below. With the sag value being known, we have

$$\frac{y_1 + y_2 - 2S}{2} = u \cosh\left(\frac{x_1 + x_2 - 2v}{2u}\right) + \beta. \tag{3.1}$$

Subtracting equation (1.2) from (1.1) gives $f(u, v)$ and subtracting equation (3.1) from (1.1) gives $g(u, v)$ as follows:

$$f(u, v) = \cosh\left(\frac{x_2 - v}{u}\right) - \cosh\left(\frac{x_1 - v}{u}\right) - \frac{y_2 - y_1}{u} = 0$$

$$g(u, v) = \cosh\left(\frac{x_1 - v}{u}\right) - \cosh\left(\frac{x_1 + x_2 - 2v}{2u}\right) - \frac{y_1 - y_2 + 2S}{2u} = 0$$

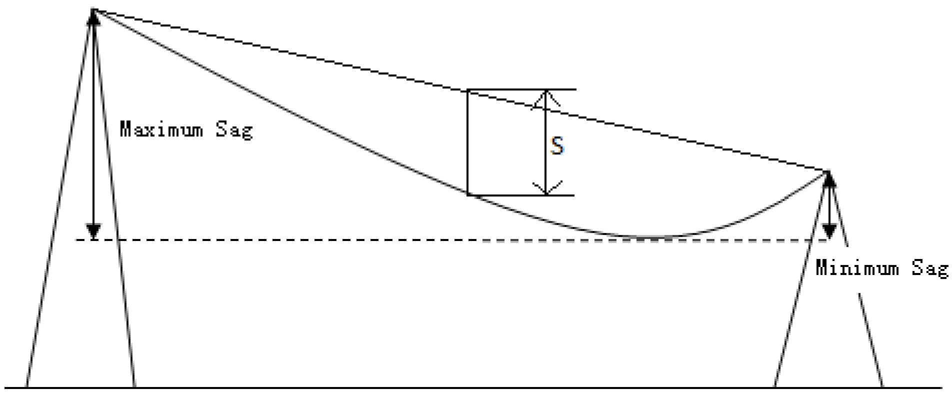

Figure 4: Sag value at middle is known

Differentiating f and g with respect to u and v gives

$$f_u' = -\frac{1}{u^2}\left[\sinh\left(\frac{x_2-v}{u}\right)(x_2-v) - \sinh\left(\frac{x_1-v}{u}\right)(x_1-v) - (y_2-y_1)\right]$$
$$f_v' = -\frac{1}{u}\left[\sinh\left(\frac{x_2-v}{u}\right) - \sinh\left(\frac{x_1-v}{u}\right)\right]$$
$$g_u' = -\frac{1}{u^2}\left[\sinh\left(\frac{x_1-v}{u}\right)(x_1-v) - \sinh\left(\frac{x_1+x_2-2v}{2u}\right)\left(\frac{x_1+x_2-2v}{2}\right) - \left(\frac{y_1-y_2+2S}{2}\right)\right]$$
$$g_v' = \frac{1}{u}\left[\sinh\left(\frac{x_1+x_2-2v}{2u}\right) - \sinh\left(\frac{x_1-v}{u}\right)\right]$$

Accordingly, we can use equation (1.4) to solve u and v, noting that the third unknown β can be solved using equation (1.1).

4 Representation of catenaries in CAD system

In most CAD systems, B-spline curves and surfaces are the de facto standard for modeling and representing complex shapes that cannot be described by line and arc. Since a B-spline curve is a piecewise rational curve, approximation is needed when it is used to represent a non-rational curve such as a catenary. In this section we discuss how to create a B-spline curve that approximates a given catenary curve within any given tolerance.

Both hyperbolic sine and cosine have the following convergent Taylor series (or Maclaurin series) for all real values of x.

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \tag{4.1}$$
$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \tag{4.2}$$

An infinite series cannot be represented in a computer. It has to be truncated and represented as a polynomial. Accordingly, the truncated polynomial curve can be represented as a B-spline curve in CAD systems. Before doing so, however, we need to consider the following two aspects:

- Does the series converge to its analytic function uniformly? By observation it is readily seen that it converges faster when x is at the neighborhood of 0 and slower when x is large. This in turn means that we would need a lower degree polynomial for x near 0 and higher degree polynomial when x is away from 0, which is impractical.
- Does the truncated polynomial have the best approximation to its analytic function?

These two obstacles lead us to look for the Legendre series which, by the following theorem, has a promising property in approximation theory.

Theorem 4.1 *Let E_ρ be an ellipse on the complex plane with the center at $(0,0)$, the major semi-axis, $a = (\rho + 1/\rho)/2$, on the axis of real numbers, and the minor semi-axis, $b = (\rho - 1/\rho)/2$, on the axis of imaginary numbers. Let $f(z)$ be analytic in the interior of E_ρ with $\rho > 1$, but not in the interior of any $E_{\rho'}$ with $\rho' > \rho$. Then,*

$$f(z) = \sum_{k=0}^{\infty} a_k l_k(z)$$

where $l_k(z)$ are Legendre polynomials of degree k and

$$a_k = \frac{2k+1}{2} \int_{-1}^{+1} l_k(x) f(x) dx$$

That is to say there exists a Legendre series converging absolutely and uniformly to $f(z)$ on any closed set in the interior of E_ρ. Moreover,

$$\lim_{k\to\infty} \sup |a_k|^{1/k} = \frac{1}{\rho}.$$

Proof of the above theorem may be found in [3] and the computation of a_k can be found in [4],[5].

In real analysis, we are concerned with analytic functions f on a real interval $[-1,1]$ (If f is defined in an arbitrary interval $[a,b]$, we can reparametrize $f(x)$ by $t = (2x - b - a)/(b - a)$ such that $t \in [-1,1]$). Then, by the Neumann theorem, there exists a Legendre series converging absolutely and uniformly to f on $[-1,1]$. Further, the converging speed is determined by ρ. Therefore, problems with respect to the use of a Taylor series are avoided if a Legendre series is used.

Since the Legendre series converges absolutely and uniformly to $f(x)$ on $[-1,1]$. Therefore, given an arbitrarily small number $\varepsilon > 0$, there exists M such that when $m > M$ we have

$$\left| f(x) - \sum_{k=0}^{m} a_k l_k(x) \right| < \varepsilon.$$

Accordingly, we can use the truncated Legendre series to approximate $f(x)$. It is only a matter of polynomial basis conversion to convert the truncated Legendre series to the B-spline form[6].

References

[1] http://en.wikipedia.org/wiki/Catenary

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