Least Squares Approximation by Circle

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1 Approximation without constraints

We want to find a circle that fits the given set of points best in a sense of least squares approximation. Let a circle be represented as

$$x^2 + y^2 + 2Ax + 2By + C = 0 \tag{1-1}$$

Then, the center of the circle is $(-A, -B)$ and the radius is $r = \sqrt{A^2 + B^2 - C}$. Assume this circle is used to approximate the given set of points $\mathbf{p}_i$ $(i = 1, 2, \cdots, n)$. Then, the squared error with respect to $\mathbf{p}_i = (x_i, y_i)$ is $(x_i^2 + y_i^2 + 2Ax_i + 2By_i + C)^2$. Accordingly, the total squared errors are given by

$$\phi = \sum_{i=1}^{n} (x_i^2 + y_i^2 + 2Ax_i + 2By_i + C)^2.$$

We thus want to find A, B, C such that ϕ is minimized, which is equivalent to solving the following system of linear equations:

$$\frac{\partial \phi}{\partial A} = 0, \quad \frac{\partial \phi}{\partial B} = 0, \quad \frac{\partial \phi}{\partial C} = 0.$$

Explicitly, we need to solve

$$2\sum_{i=1}^{n} x_i^2 A + 2\sum_{i=1}^{n} x_i y_i B + \sum_{i=1}^{n} x_i C + \sum_{i=1}^{n} (x_i^2 + y_i^2) x_i = 0$$
$$2\sum_{i=1}^{n} x_i y_i A + 2\sum_{i=1}^{n} y_i^2 B + \sum_{i=1}^{n} y_i C + \sum_{i=1}^{n} (x_i^2 + y_i^2) y_i = 0$$
$$2\sum_{i=1}^{n} x_i A + 2\sum_{i=1}^{n} y_i B + nC + \sum_{i=1}^{n} (x_i^2 + y_i^2) = 0$$

For numerical stability it is recommended to transform the given set of points to the local coordinate system whose origin is at the center of gravity of n points.

2 Approximation with two constraints

Referring to Figure 1, a planar plate has been cut by a circular cutter. In CAD system, such cut is modeled by a Boolean intersection between a cylinder and the plate. Ideally the intersection curve should be a circular arc. Depending on numerical noise and/or the solid modeler used for Boolean operations, however, it may be approximated by a B-spline curve (often a cubic polynomial B-spline curve).

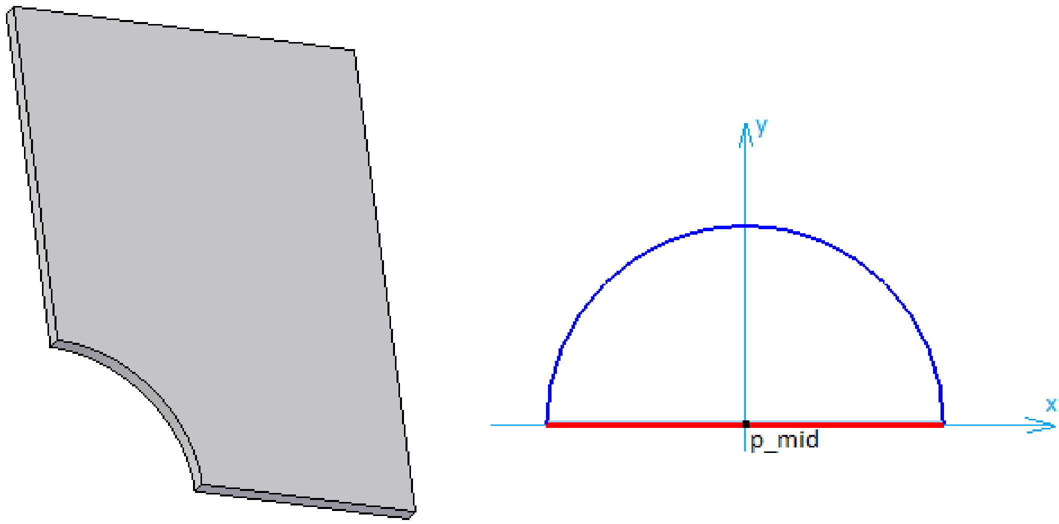

Figure 1: A plate cut by circular cutter

When this computer generated model is sent to workshop for manufacture, the first thing a CAD/CAM system needs to do is to generate cutting paths for numerical controlled (NC) machines. Therefore, it is desirable to represent the cutting path by a circular arc. One possible approach is to stroke the B-spline curve to obtain enough sampling points and then apply the least squares approximation outlined in the previous section to cerate the best fit circle. Since it is essential to have continuous NC paths, the best fit circle needs to interpolate the endpoints of the B-spline curve. Unfortunately, this cannot be guaranteed by the approach discussed previously. We need to explore a different approach.

In this section, we discuss how to create a circle that interpolates the endpoints. To simplify the equation and minimize any numerical noise, we first compute the mid-point of the chord and then transform the given points to the new coordinates system whose origin is at the mid-point and the x-axis is parallel to the chord (referring to Figure 1). After transformation, the given points are symmetric to the y-axis and hence A in equation (1-1) is zero. Accordingly, we have

$$x^2 + y^2 + 2By + C = 0 \qquad (1-2)$$

It is also known that y_1 and y_n are zeros in the new coordinate system. Therefore, $C = -x_1^2$ if equation (1-2) holds at (x_1, y_1) (i.e., the circle interpolates the endpoints). Our task boils down to find B so that the circle best fits the remaining points in a least squares sense. Let

$$\phi(B) = \sum_{i=1}^{n} \left(x_i^2 + y_i^2 + 2y_iB + C\right)^2$$

To minimize the summation of squared errors is equivalent to solving $\dfrac{d\phi}{dB} = 0$, i.e.,

$$2\sum_{i=1}^{n} \left(x_i^2 + y_i^2 + 2y_iB + C\right) y_i = 0.$$

The above equation is met if

$$B = -\frac{\sum_{i=1}^{n}(x_i^2 + y_i^2 + C)y_i}{2\sum_{i=1}^{n} y_i^2}.$$

Consequently, the center of circular arc is $(0, -B)$ and the radius is $\sqrt{B^2 - C}$. It should be pointed out that the obtained center of circle is defined in the new coordinate system. Therefore, we need to transform it back to the original coordinate system.

In this section we presented a simple example where the B-spline curve can be approximated by a single circular arc. For free form curve and surface modeling, a B-spline curve may not be approximated by a single circular arc. In this case, one may need to compute a set of lines and arcs to approximate the curve as discussed in the listed publications.

References

[1] Bolton, K.M. (1975) Biarc curves, *Computer Aided Design* Vol. 7, 89-92.

[2] Meek, D.S and Walton, D.J. (1993) Approximating quadratic NURBS curves by arc splines, *Computer Aided Design* Vol. 25, 371-376.